Recent Advances in RESTART Simulation

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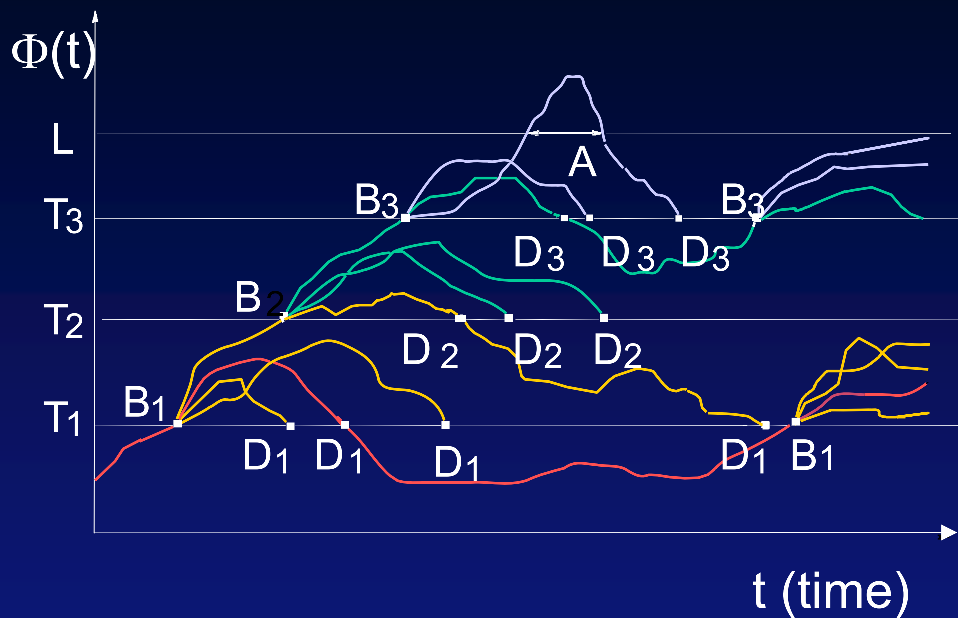
CONTENTS

- Variance of the estimator
- Importance function for Jackson networks
- Importance function for highly-dependable systems
- Conclusions

Description of RESTART (I)



$$P = \Pr\{A\} = \Pr\{\Phi \geq L\}$$

$$\Pr\{C_i\} = \Pr\{\Phi \geq T_i\}$$

R_i: Number of trials at B_i

$$r_i = \prod_{j=1}^{i} R_j$$

$$C_1 \supset C_2 \supset C_3 \supset \ldots C_M \supset A$$

$$P\{A\} = P\{C_1\} \bullet P\{C_2 / C_1\} \bullet \ldots \bullet P\{A/C_M\}$$

$$\hat{P} = \frac{N_A}{r_M \cdot N}$$

N : No. of simulated events (retrials not included)

N_A : No. of events A (retrials included)

Description of RESTART (II)

$$P = \Pr\{A\} = \Pr\{Q_2 \geq L\}$$

$$\Pr\{C_i\} = \Pr\{\Phi \geq T_i\}$$

$$\Phi = \frac{\ln \rho_2}{\ln \rho_1} Q_1 + Q_2$$

$$A \not\subset C_i \;\; i = 1, \ldots, M \;\text{ then: } P\{A\} = \sum_{i=1}^{M} P\{C_i\} P\{A / C_i\}$$

$$\hat{P} = \sum_{i=1}^{M} \frac{N_{Ai}}{r_i \cdot N}$$

N : No. of simulated events (retrials not included)

N_{Ai} : No. of events A in retrials from sets C_i

Variance of the estimators

- **Case** $A \subset C_M$:

$$V(\hat{P}) = \frac{K_A P}{N}\left[\frac{1}{r_M} + \sum_{i=1}^{M} \frac{s_i P_{A/i}\left(R_i - 1\right)}{r_i}\right]$$

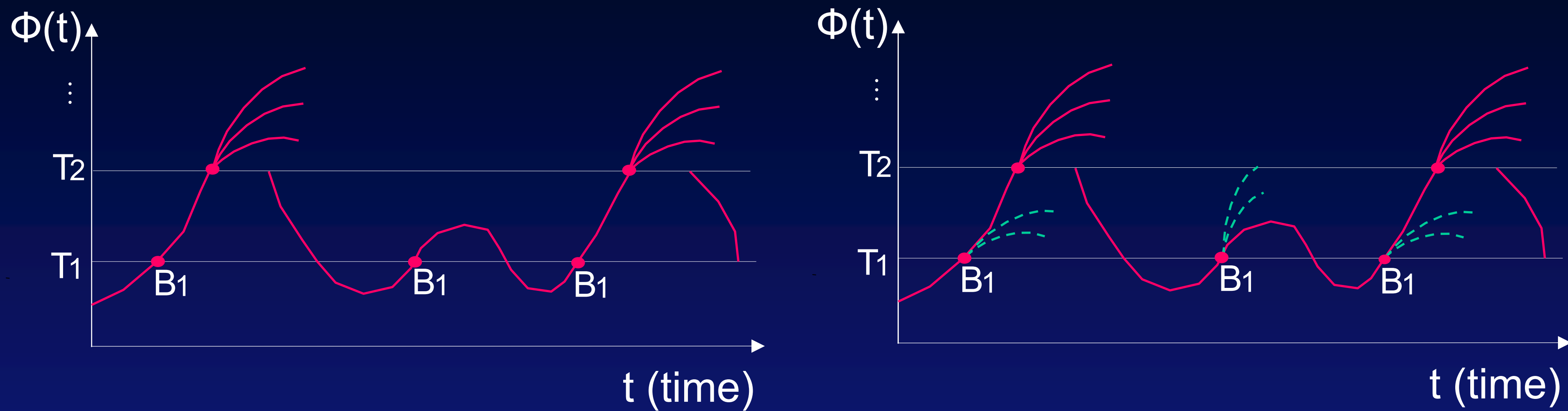
With $K_A = V\left(N_A^0\right) / E\left[N_A^0\right]$

- **General case:**

$$V(\hat{P}) = \sum_{i=0}^{M} \frac{K_{Ai} P_{Ai}}{N r_i} + \frac{K_A P}{N} \sum_{i=1}^{M} \frac{s_i P_{A/i}\left(R_i - 1\right)}{r_i}$$

With $K_{Ai} = V\left(N_{Ai}^0\right) / E\left[N_{Ai}^0\right]$

Proof



M-1 thresholds (T_2 to T_M)   M thresholds (T_1 to T_M)

➢ **Proof by induction:**

- Crude simulation $\Rightarrow$ Monothreshold simulation
- M-1 threshold simulation $\Rightarrow$ M threshold simulation

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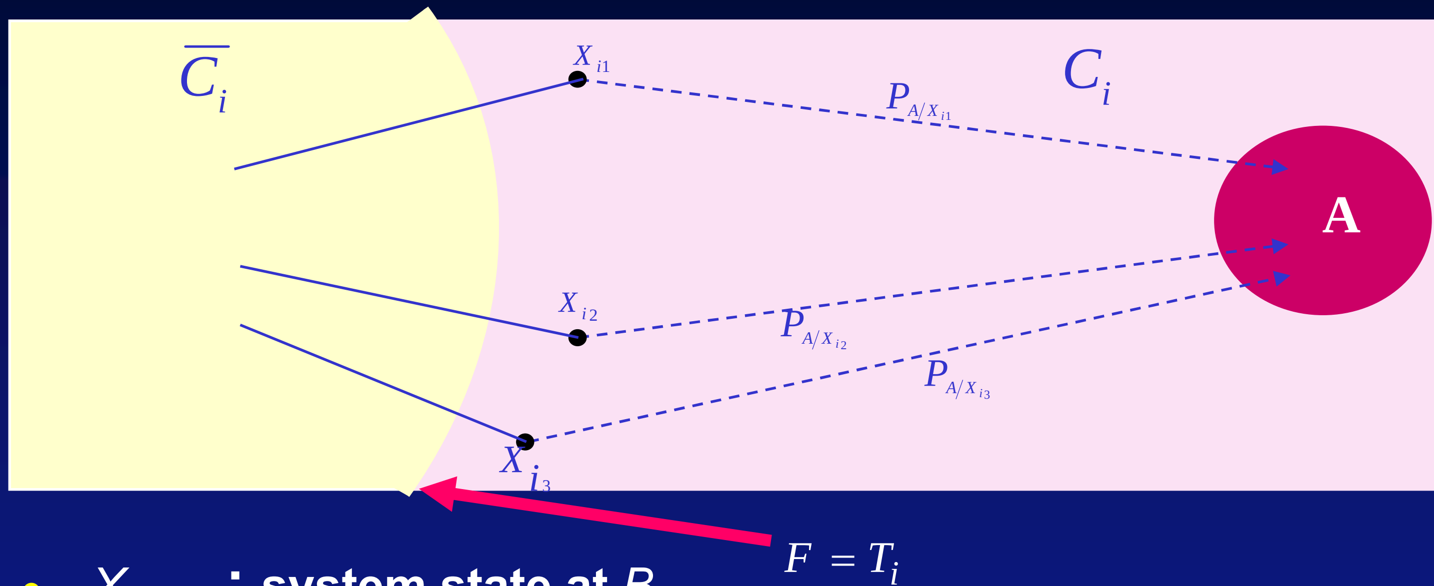
Factor f_v (I)

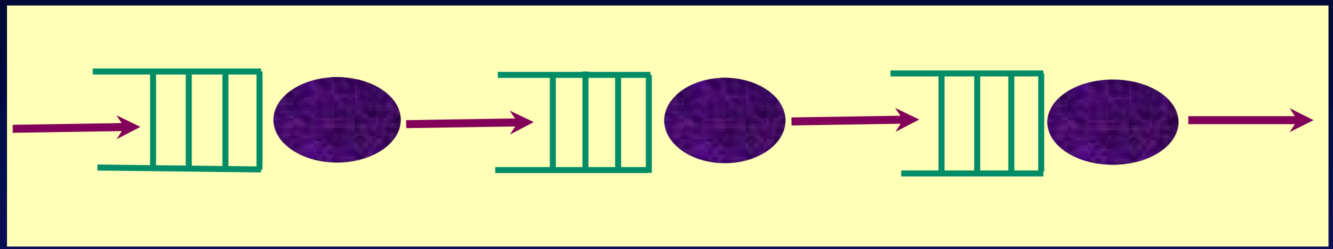
$$f_v \le Max(s_i) \qquad s_i = \frac{a_i}{K_A}\left[K'_i + \frac{V(P^*_{A/X_i})}{(P^*_{A/i})^2}\gamma_i\right] \square\, 1 + \frac{V(P^*_{A/X_i})}{(P^*_{A/i})^2}$$



- X_i : system state at B_i
- P^*_{A/X_i} : importance of state X_i
- $P^*_{A/i}$: expected value of P^*_{A/X_i}
- γ_i : factor reflecting the autocovariance of P^*_{A/X_i}

Importance Function (I)

- **Three-queue Jackson tandem network**



  - Importance function (If $\rho_1 > \rho_2 > \rho_3$):

$$\Phi = Q_1 \frac{\ln \rho_1}{\ln \rho_3} + Q_2 \frac{\ln \rho_2}{\ln \rho_3} + Q_3$$

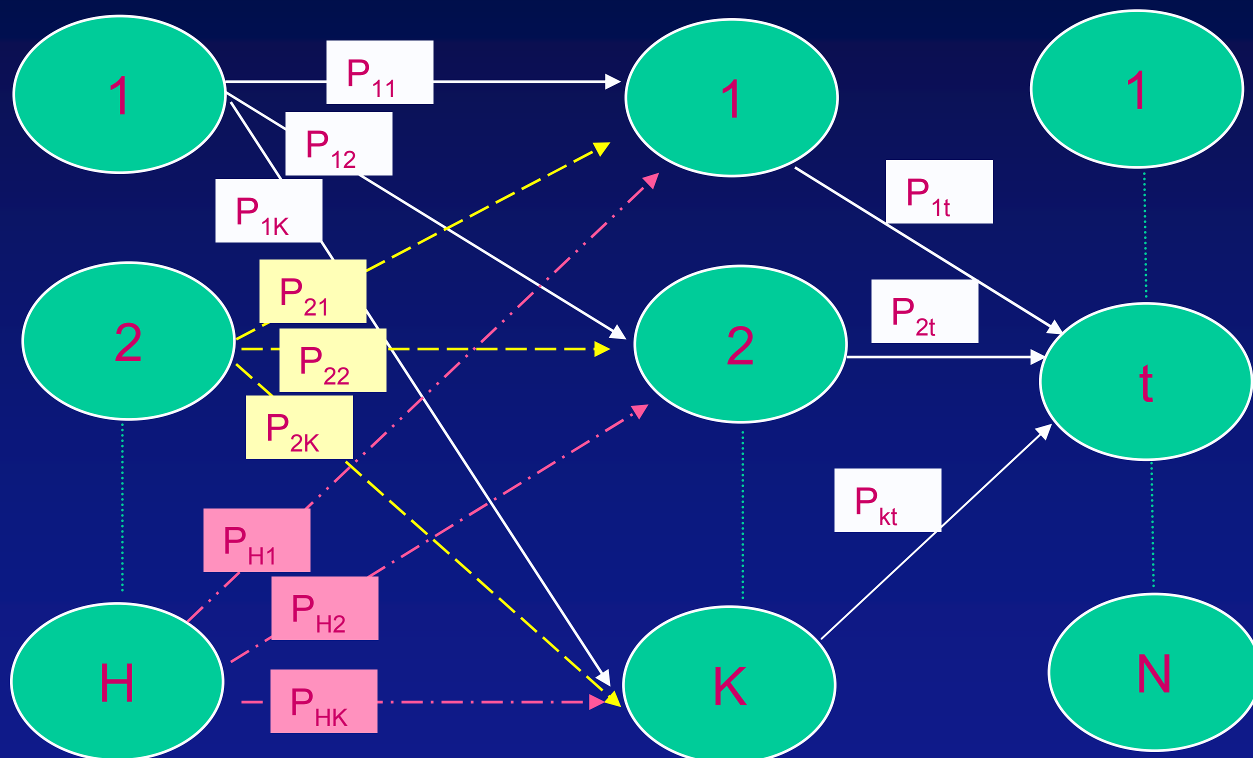
If $\rho_1 < \rho_2 < \rho_3$, or if $\rho_2 < \rho_1 < \rho_3$, $\Phi = Q_1 + Q_2 + Q_3$

If $\rho_2 < \rho_3 < \rho_1$, $\Phi = Q_1 \frac{\ln \rho_1}{\ln \rho_3} + Q_2 + Q_3$

If $\rho_1 < \rho_3 < \rho_2$, or if $\rho_3 < \rho_1 < \rho_2$, $\Phi = \frac{\ln \rho_2}{\ln \rho_3}(Q_1 + Q_2) + Q_3$

Factor f_v (III)

- *Example 2:* ***A Jackson network with H nodes in the first stage K nodes in the second stage and N nodes in the third stage.***



Importance Function (II)
General Jackson networks

$$P = \Pr\left\{Q_t \geq L\right\}$$

$$\Phi = \sum_{i=1}^{H} \alpha_{1i} \frac{\ln\left(\rho_t / \rho_{ti}^{*}\right)}{\ln \rho_t} + \sum_{j=1}^{K} \alpha_{2j} \frac{\ln\left(\rho_t / \rho_{tj}^{\perp}\right)}{\ln \rho_t} + Q_t$$

$$\rho_t = \frac{\gamma_t + \sum_{j=1}^{K} \lambda_j p_{jt}}{\mu_t} = \frac{\lambda_t}{\mu_t} \qquad \rho_{ti}^{*} = \frac{\gamma_t + \sum_{j=1}^{K}\left[\lambda_j + (\mu_i - \lambda_i)\left(p_{ij} + \sum_{l \neq i} p_{il} p_{lj}\right)\right] p_{jt}}{\mu_t}$$

$$\alpha_{1i} = q_{1i}\left(1 + \frac{\sum_{l \neq i} \gamma_l \sum_{j=1}^{K} p_{lj} p_{jt} + \sum_{j=1}^{K} \gamma_j p_{jt} + \gamma_t}{\mu_{1i} \sum_{j=1}^{K} p_{ij} p_{jt}}\right); \quad \alpha_{2j} = q_{2j}\left(1 + \frac{\sum_{i=1}^{H} \gamma_i \sum_{l \neq j} p_{il} p_{lt} + \sum_{l \neq j} \gamma_l p_{lt} + \gamma_t}{\mu_j p_{jt}}\right)$$

Simulation Results (I)

- ***Example 1:*** *Jackson Network with 7 nodes arrival rate from exterior* $\gamma_i = 1$; $i = 1, \ldots, 7$

Transition Probability Matrix

	1	2	3	4	5	6	t	Ext.
1	0.2	0.2	0.2	0.2	0	0	0	0.2
2	0.2	0.2	0.2	0.2	0	0	0	0.2
3	0.1	0.1	0.1	0.1	0.2	0.2	0	0.2
4	0.1	0.1	0.1	0.1	0.2	0.2	0	0.2
5	0.1	0.1	0.1	0.1	0	0.1	0.3	0.2
6	0.1	0.1	0.1	0.1	0.1	0	0.3	0.2
t	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2

Simulation Results (II)

- *Example 1:* *Jackson network with 7 nodes. Nodes 1 and 2 are connected with the target node through 2 intermediate nodes (distance 3).*

$$P(Q_t \geq 70) = 8.8 \cdot 10^{-35}; \qquad \rho_t = 0.3262; \quad \Phi = a\sum_{i=1}^{2} Q_i + b\sum_{j=3}^{4} Q_j + c\sum_{k=5}^{6} Q_k + Q_t$$

ρ_1	ρ_2	ρ_2	a	b	c	Events (millions)	Time (minutes)	Gain (events)	f_V
0.62	0.51	0.41	0	0.32	0.55	35	8.6	1.5 10^{29}	3.2
"	"	"	0.07	0.32	0.55	31	7.8	1.7 10^{29}	2.8
"	"	"	0	0	0.55	2120	482	2.5 10^{27}	190
0.47	0.51	0.41	0	0.32	0.55	21	5.8	2.5 10^{29}	1.9
"	"	"	0.1	0.32	0.55	13	3.6	4.1 10^{29}	1.2
"	"	"	0	0	0.55	614	206	2.6 10^{27}	55
0.31	0.31	0.31	0	0.52	0.70	13	3.6	4.1 10^{29}	1.2
"	"	"	0.16	0.52	0.70	12	3.3	4.4 10^{29}	1.1
"	"	"	0	0	0.70	26	6.1	2.0 10^{29}	2.4

Simulation Results (III)

Example 2: Large Jackson network with 15 nodes, four of them at distance 3 from the target node. The target node has the lowest load.

$$P\left(Q_t \geq 70\right) = 1.6 \cdot 10^{-33}; \qquad \rho_t = 0.34; \quad \Phi = 0\sum_{i=1}^{4} Q_i + b\sum_{j=5}^{9} Q_j + c\sum_{k=10}^{14} Q_k + Q_t$$

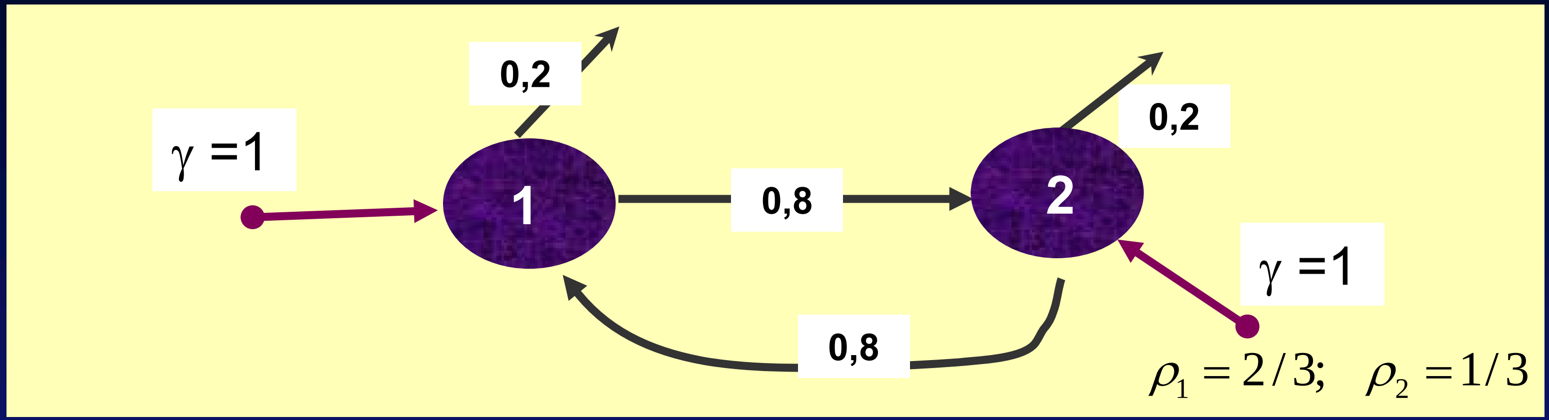
- 20 minutes of computational time for estimating this probability with a relative error of 10%.

$$P\left(Q_t \geq 30\right) = 8.8 \cdot 10^{-15}$$

- 4 minutes of computational time for estimating this probability with the same importance function and the same relative error.

Simulation Results (IV)

- *Example 3: Network with strong feedback: 2-node Jackson network.*



$P(Q_2 \geq 70) = 4.0 \cdot 10^{-34}$; $\Phi = 0.369 Q_1 + Q_2$

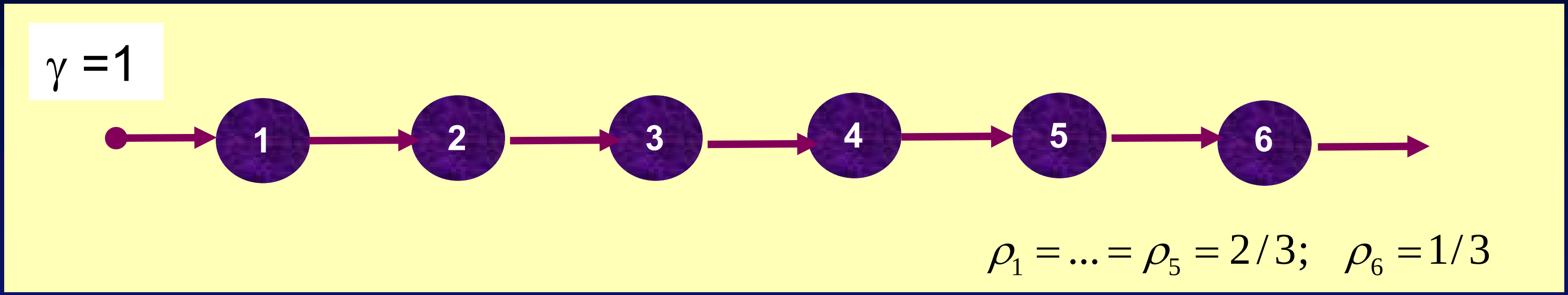
- 30 minutes of computational time for estimating this probability with a relative error of 10%.

$P(Q_2 \geq 30) = 4.9 \cdot 10^{-15}$;

- 4 minutes of computational time for estimating this probability with the same importance function and the same relative error.

Simulation Results (V)

- *Example 4: Network with high dependencies and nodes at distance greater than 2: 6-node Jackson tandem network.*



$$P\left(Q_t \geq 30\right) = 4.9 \cdot 10^{-15}; \qquad \Phi = 0.369 Q_4 + 0.369 Q_5 + Q_6$$

- 30 minutes of computational time for estimating this probability with a relative error of 10%.

$$\Phi = 0.15 Q_1 + 0.2 Q_2 + 0.25 Q_3 + 0.31 Q_4 + 0.37 Q_5 + Q_6;$$

- 10 minutes of computational time for estimating this probability with the same relative error.

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Importance function (I)

Objective: **All the system states of the same threshold, X_i, must have similar importance.**

- **In a general system there are minimal cutsets with different cardinalities.**

- **It is more probable that system failure will be due to the failure of all the components of a minimal cutset with the lowest cardinality.**

- **The "distance" to the system failure is related with the number of components that remain operational in the cutset with the lowest number of operational components.**

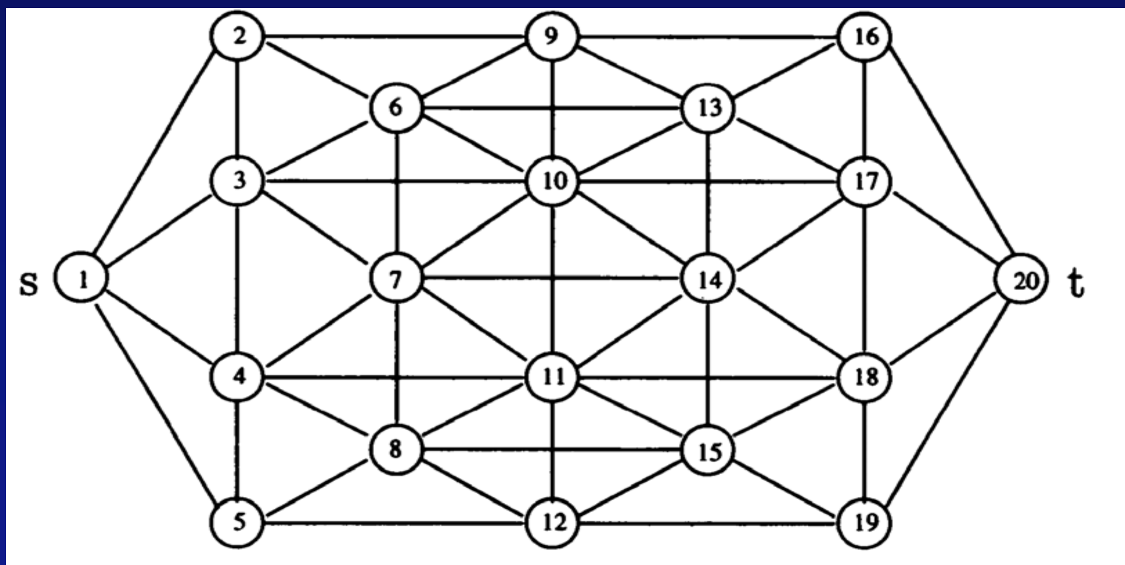
  - **Importance function: $\Phi(t)$ = cl – oc(t)**

Importance function (II)

Network with redundancies

- **Importance function:** $\Phi(t) = \text{cl} - \text{oc}(t)$

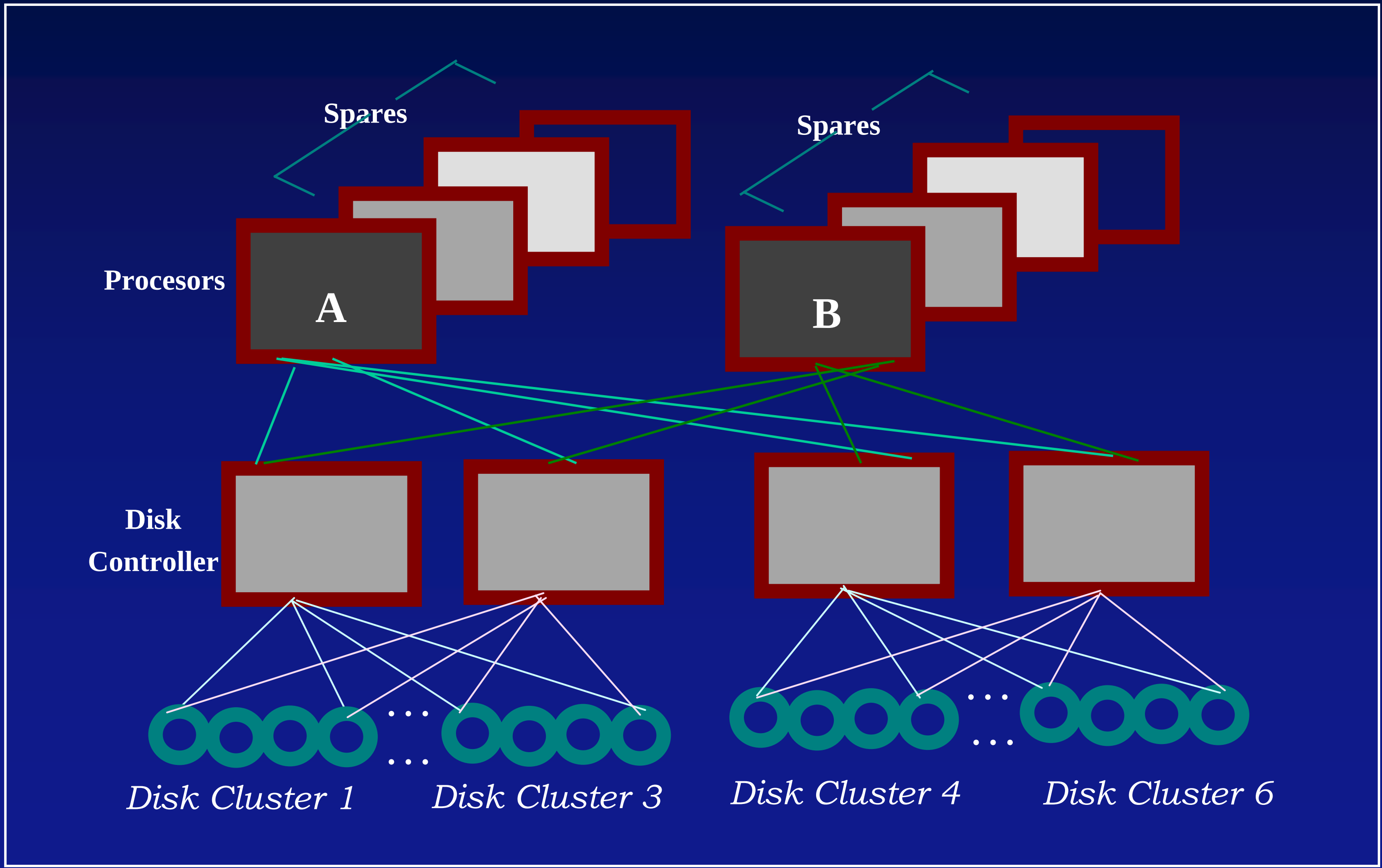
***Simulation* 83 (12), 2007, 821-828.**



Importance function (III)

- *Computing system*
  - **Importance function: $\Phi(t)$ = cl – oc(t)**

Simulation 83 (12), 2007, 821-828.



First model

CONSECUTIVE-*K*-OUT-OF-*N* : F SYSTEM (or C(*k*, *n*: F) system) (Kontoleon, 1980)

- **A sequence of *n* ordered components along a line such that the system fails if and only if at least *k* consecutive components in the system have failed.**
- **All the components have the same probability to fail**
- **The component lifetimes are exponentially distributed**
- **There is not reparation service**

Other Models

- C(*k*, *n*: F) REPARAIBLE SYSTEM **AND (*k*-1)-STEP MARKOV DEPENDENCE**

- **Lam and Ng, 2001:**

Component lifetimes and repair times exponentially distributed (Markovian model)

- **Xiao et al. 2007:**

Component lifetimes exponentially distributed

Repair times: general distribution. Only one reparation service.

Rare event simulation: Importance Sampling; Conditional **expectation.**

- **This paper (2008):**

Component lifetimes and repair times: general distribution.

Rare event simulation: RESTART

Model Features

Non-Markov C(*k*, *n*: F) REPARAIBLE SYSTEM **AND (*k*-1)-STEP MARKOV DEPENDENCE**

- **The system fails if and only if *k* or more consecutive components have failed.**
- **The lifetime of components and the repair time have a general distribution.**
- **A failed component has the highest repair priority if the system failure risk after repair of that component is lower than after repair of any other failed component.**
- **If the system has failed, then no more components will fail.**
- **If there are *h* (*h* < *k*) consecutive failed components that precede the component *i*, the residual lifetime of component *i* will have increasing failure rate as *h* increases,**

Simulation Results (I)

- ***Table 1:*** ***Unreliability estimates for C(4, 60: F) system. 95% confidence interval = ± 10%***

Model	Interval $(0, t_e)$	*Unrelia-bility*	Run-time (seconds)	Gain in time	Factor f_T	Factors $f_O \times f_V$
EL A	(0, 25)	3.4×10^{-6}	3	4.8×10^{1}	2.6	1.3
EL A	(0, 5)	3.8×10^{-8}	6	7.7×10^{3}	4.9	2.1
EL A	(0, 1)	1.2×10^{-10}	24	4.9×10^{5}	12.5	2.4
WL A	(0, 25)	8.6×10^{-13}	697	2.1×10^{8}	28.6	2.4
WL A'	(0, 25)	8.3×10^{-6}	11	1.6×10^{1}	2.4	2.0
WL A'	(0, 1)	2.0×10^{-10}	149	3.8×10^{5}	11.0	2.2

- **Repair times:** Lognormal in all the models.
- **Lifetimes:** Exponencial (models EL) or Weibull (models WL).
- **Model WL A':** Componets are 640 u.t. old

Simulation Results (II)

- ***Table 2:*** ***Unavailability and MTBF estimates for C(4, 60: F) system. 95% confidence interval = ±10%***

Model	Unavaila-bility	MTBF	Run-time (seconds)	Factor f_T	Gain in time	Factors $f_V x f_O$
EL A	1.6×10^{-5}	5.8×10^{5}	4	2.6	38	4.4
EL B	4.6×10^{-8}	2.8×10^{8}	12	9.0	1267	5.9
EL C	4.2×10^{-10}	2.1×10^{9}	33	36.1	16650	7.8
WL A	2.2×10^{-5}	8.9×10^{5}	12	2.5	14	9.4
WL B	8.8×10^{-8}	1.4×10^{8}	45	8.2	263	17.7
WL C	5.6×10^{-10}	1.7×10^{9}	194	38.7	3159	29.4

- **Repair times: Lognormal** in all the models.
- **Lifetimes: Same mean in Exponencial and Weibull models.**
- **Models B, C: Mean lifetimes 3 and 10 times greater than model A**

Conclusions

The paper has provided:

- Formulas for the variance of the estimator in the general case where the rare set can be reached from any threshold.
- Analysis of some “difficult” networks showing the efficiency of RESTART for simulating most (may be all) Jackson networks with the given importance function.
- Analysis of some highly-dependable systems showing the efficiency of RESTART for simulating different types of systems with the same importance function.

In the future:

- Study the possibility of apply these formulas to other systems, given that RESTART is not so dependent as other methods on particular features of the model.