Reversibility-Based Construction of Efficient IS Schemes for Simulating Jackson Networks *

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Outline

- Motivation

- Model and Notation

- State-Dependent Importance Sampling

- Approximating the Zero-Variance C.O.M.
  - Adaptive IS Heuristics (de Boer et al.)

- C.O.M. to Approach the Most Likely Path
  - State-Dependent Heuristics (Nicola et al.)
  - Game-Based Approach (Dupuis et al.)
  - Reversibility-Based Approach (This work!)

- Experimental Results

- Conclusions and Further Research

Motivation

- Efficient simulation of rare events in queueing networks (e.g., buffer overflow)

- Applications
  - Computer Systems
  - Telecommunication Networks
  - Manufacturing, and others

- Importance sampling (IS) technique
  - State-independent (static) IS
  (does not always work!)
  - State-dependent (dynamic) IS
  (generally more effective!)

- This work: reversibility-based approach for constructing efficient dynamic IS schemes

Model and Notation

- Jackson queueing network with n nodes

  λ_i external arrival rate at Node i

  μ_i service rate at Node i

  p_{ij} routing probability from i to j

  p_{ie} routing probability from i to $exit$

  Λ_i total arrival rate at Node i
  (from traffic equations):

$$\Lambda_i = \lambda_i + \sum_{\forall j} \Lambda_j \, p_{ji}, \; i = 1, \ldots, n$$

  $\Lambda_i < \mu_i$, $i = 1, \ldots, n$ (all queues are stable!)

Model and Notation (contd.)

- Notation

  - $X_{i,t}$ $(i = 1, \ldots, n)$ number at Node i at $t \geqslant 0$

  - $\mathbf{X}_t = (X_{1,t}, \ldots, X_{n,t})$, $t \geqslant 0$, Markov process

  - $S_t = \sum_{\forall i} \beta_i \ X_{i,t}$, with $\beta_i = \{0, 1\}, i = 1, ..., n$

  When $\forall i \, \beta_i = 1$, $S_t \equiv$ network population

  When only $\beta_i = 1$, $S_t \equiv$ buffer i content

- Starting from $S_0 = 1$

  τ_K first time $S_t = K$

  τ_0 first time $S_t = 0$

  - rare event of interest:

  S_t hits K before return to 0

  - associated probability: $\gamma(K) = \mathbb{P}(\tau_K < \tau_0)$

State-Dependent (Dynamic) IS

- Generally more effective than static IS!

- Two basic approaches:
  - approximate the zero-variance C.O.M.
  - "push" the system around the most likely path (MLP) to the rare set

- Zero-variance C.O.M. and MLP:
  - not known (require the true probability!)
  - state-dependent (even for the $M/M/1$ queue!)
  - strong dependence close to boundaries (when one or more queues are empty!)
  - weak or no dependence in the interior

Approximating the Zero-Variance C.O.M

- Goal: approximate the zero-variance C.O.M.

- Methodology:
  - for each state, within a properly sized layer along the boundaries, approximate (iteratively or otherwise) the transition probabilities conditional on reaching the rare set!
  - the resulting C.O.M. is effective regardless of the initial state!

- Adaptive IS heuristics
  - cross-entropy and bio-inspired (de Boer et al. 2002, Heegaard et al. 2008)
  - stochastic approximation (Ahamed et al. 2005)
  - less effective for large networks: excessive computational effort convergence problems?

C.O.M. to Approach the MLP

- Goal: "push" the system around the most likely path (MLP) to the rare set

- Methodology:

  - determine the "optimal" C.O.M.s on key boundaries - boundaries increase with number of network nodes

  - appropriately combine these C.O.M.s - dependence range along boundaries is crucial for asymptotic efficiency!

  - the resulting C.O.M. may not be effective for initial states other than the "empty network"!

C.O.M. to Approach the MLP (contd.)

- State-dependent IS heuristics
  (Nicola and Zaburnenko 2007)

  - "push" the system along
  a few most likely trajectories to the rare set

  - effective for tandem/parallel topologies

  - applicability for large Jackson networks?

- Game-based approach
  (Dupuis, Sezer and Wang 2005-2008)

  - "push" the system along
  all possible trajectories to the rare set

  - provably effective(optimal control framework)

  - implementation and performance problems
  for large networks?

Reversibility-Based Approach

- Time-reversal argument:

  The most likely large deviation path from the initial state (empty network) to the rare set follows the reverse-time fluid-limit trajectory from the rare set to the initial state in the same network (Anantharam et al. 1990, Majewski and Ramanan 2008)

- Reverse-time fluid-limit trajectories from the rare set to the initial (empty network) state are deterministic and easy to compute!

- Goal: "push" the system along
  a few most likely trajectories to the rare set

Reversibility-Based Approach (contd.)

- Methodology:

  - identify all boundaries traversed by a few most likely paths to the rare set (from deterministic reverse-time fluid-limit trajectories)
  - determine the optimal C.O.M. on each of these boundaries (using time-reversal algorithm)
  - the C.O.M. at any state is an appropriate smoothing of the C.O.M.s on the traversed boundaries (e.g., linear/exponential weighting)

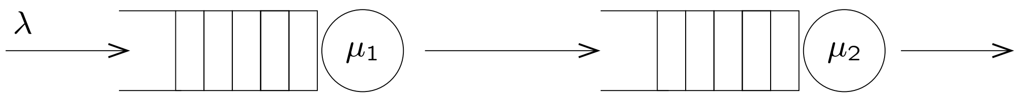
- Time-reversal algorithm (to determine the optimal C.O.M. on a given boundary):

  1. Determine the reverse-time flows in- and out- of each node from a deterministic set of n linear (traffic) equations

  2. Obtain the optimal C.O.M. such that the forward-time flows in- and out- of each node are identical to the reverse-time flows out- and in- the same node (from Step 1)

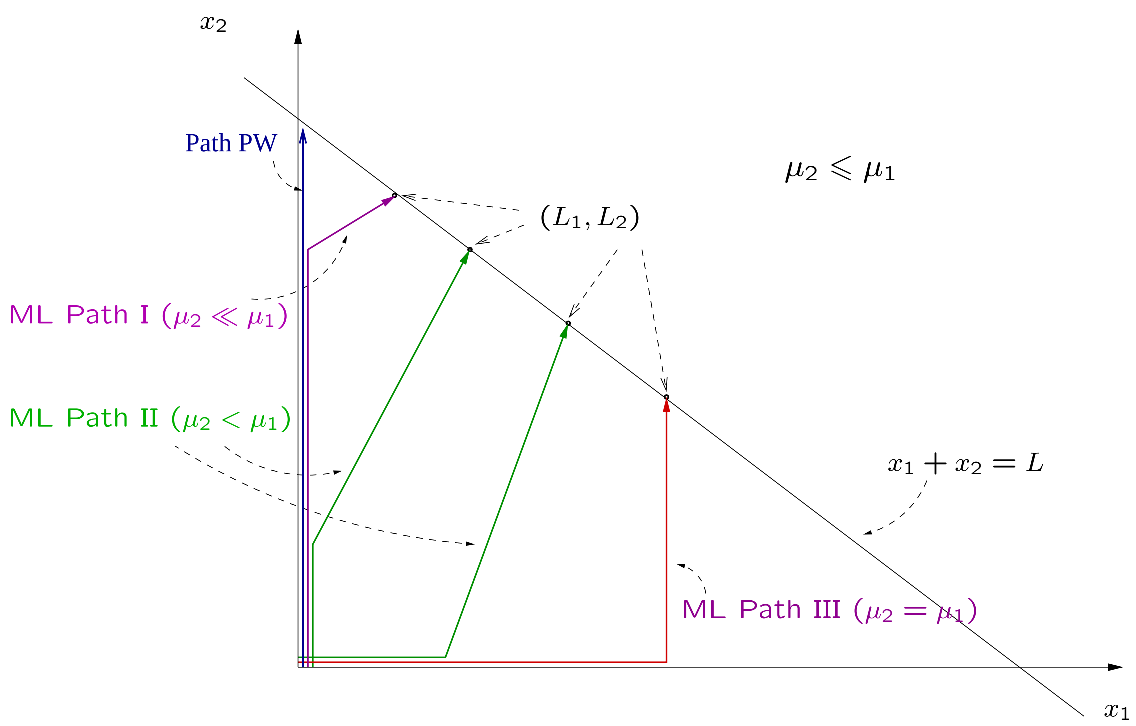
Reversibility-Based Approach (contd.)

Consider the 2-node tandem network



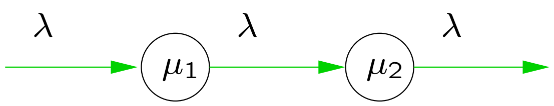
a) Forward time process

b) Reverse time process

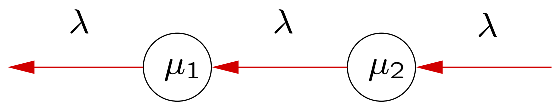


Reversibility-Based Approach (contd.)

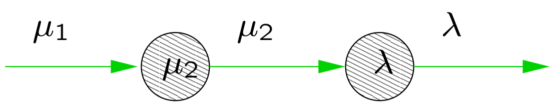
Reverse-time algorithm to determine C.O.M. on the 4 boundaries {(0,0), (1,1), (1,0), (0,1)} of the 2-node tandem network



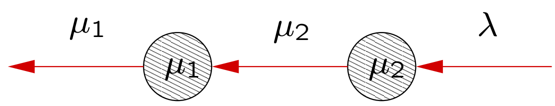
C.O.M. on boundary (0,0)



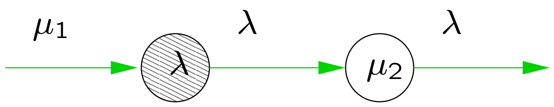
Reverse−time flows on boudary (0,0)



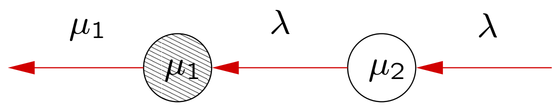
C.O.M. on boundary (1,1)



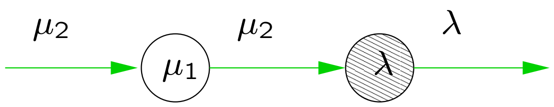
Reverse−time flows on boundary (1,1)



C.O.M. on boundary (1,0)

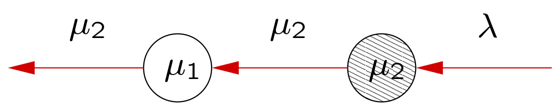


Reverse−time flows on boundary (1,0)



C.O.M. on boundary (0,1)

(only for $\mu_2 < \mu_1$!)

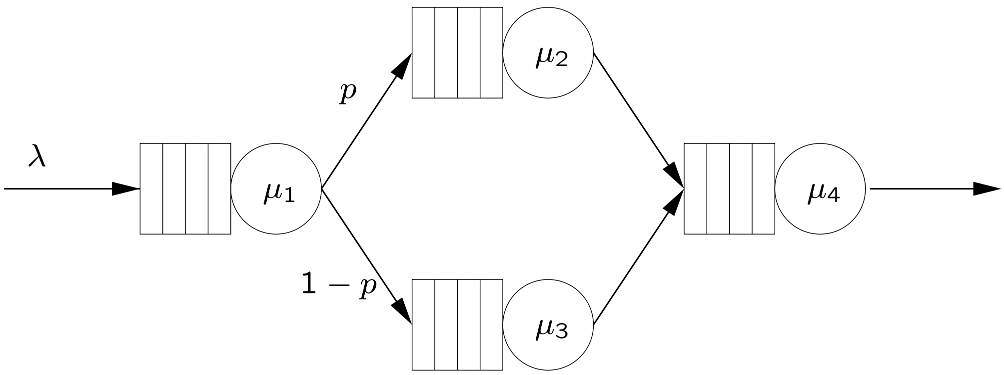


Reverse−time flows on bounday (0,1)

(only for $\mu_2 < \mu_1$!)

Experimental Results

4-node feed-forward network



N	Numerical	Simulation (IS)	
	$\gamma(N)$	b	$\hat{\gamma}(N) \pm$ RE%
10	6.2070e-03	7	6.2035e-03 $\pm$ 0.08
25	*	6	6.4827e-09 $\pm$ 0.23
50	*	7	6.0089e-20 $\pm$ 0.36
100	*	8	6.4318e-43 $\pm$ 0.58

A Symmetric 4-Node Feed-Forward Network

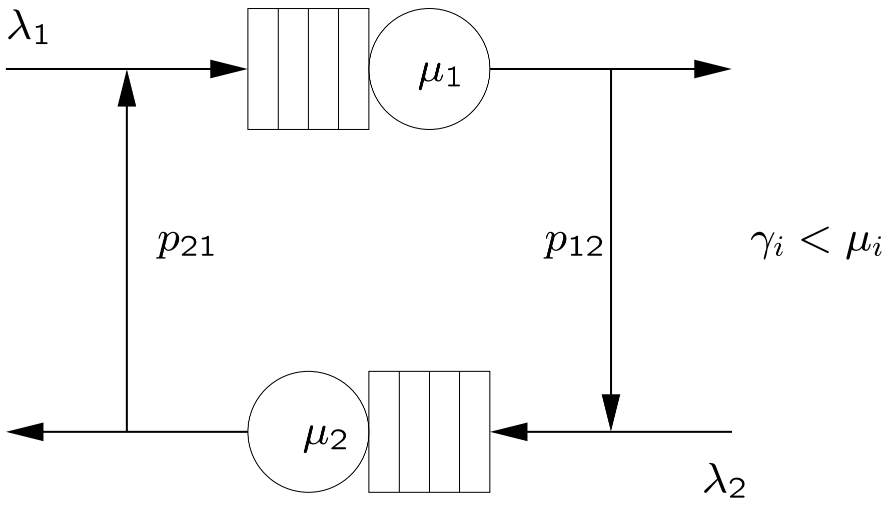
$(\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0.33)$

$\lambda = 0.1, \mu_1 = \mu_4 = 0.3, \mu_2 = \mu_3 = 0.15$

$p = 0.5$

Experimental Results (contd.)

2-node feedback network



N	Numerical	Simulation (IS)	
	$\gamma(N)$	b	$\widehat{\gamma}(N) \pm$ RE%
25	6.9823e-07	7	6.9762e-07 $\pm$ 0.20
50	4.3014e-14	8	4.2891e-14 $\pm$ 0.23
100	7.7647e-29	10	7.7717e-29 $\pm$ 0.33

A Symmetric 2-Node Feedback Network

$(\rho_1 = \rho_2 = 0.5)$

$\lambda_1 = \lambda_2 = 0.1, \mu_1 = \mu_2 = 0.4$

$p_{12} = p_{21} = 0.5$

Conclusions and Further Research

- Reversibility-based approach to dynamic IS

  - applicable to any Jackson network

  - construction is simple and straightforward

  - fewer trajectories $\Rightarrow$ less boundaries $\Rightarrow$ less construction overhead and better performance

  - preliminary results indicate asympt. efficiency

- Further research

  - narrow down the set of likely trajectories

  - impact of smoothing and boundary thickness on asymptotic and actual performance

  - automated tools and empirical studies

  - proof of asymptotic efficiency (via LD theory)

  - extensions to non-Jackson queueing networks