Optimal reinsurance with ruin probability target

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Ruin, solvency and reinsurance

"*reinsurance plays an important role in reducing the risk in an insurance portfolio*."

Goovaerts & Vyncke (2004). Reinsurance Forms *in* Encyclopedia of Actuarial Science.

"*reinsurance is able to offer additional underwriting capacity for cedants, but also to reduce the probability of a direct insurer's ruin* ."

Engelmann & Kipp (1995). Reinsurance. *in* Encyclopaedia of Financial Engineering and Risk Management.

Proportional Reinsurance (Quota-Share)

- claim loss X : αX paid by the cedant, $(1-\alpha)X$ paid by the reinsurer,
- premium P : αP kept by the cedant, $(1-\alpha)P$ transfered to the reinsurer,

Nonproportional Reinsurance (Excess-of-Loss)

- claim loss X : $\min\{X, u\}$ paid by the cedant, $\max\{0, X-u\}$ paid by the reinsurer,
- premium P : P_u kept by the cedant, $P-P_u$ transfered to the reinsurer,

Proportional versus nonproportional reinsurance

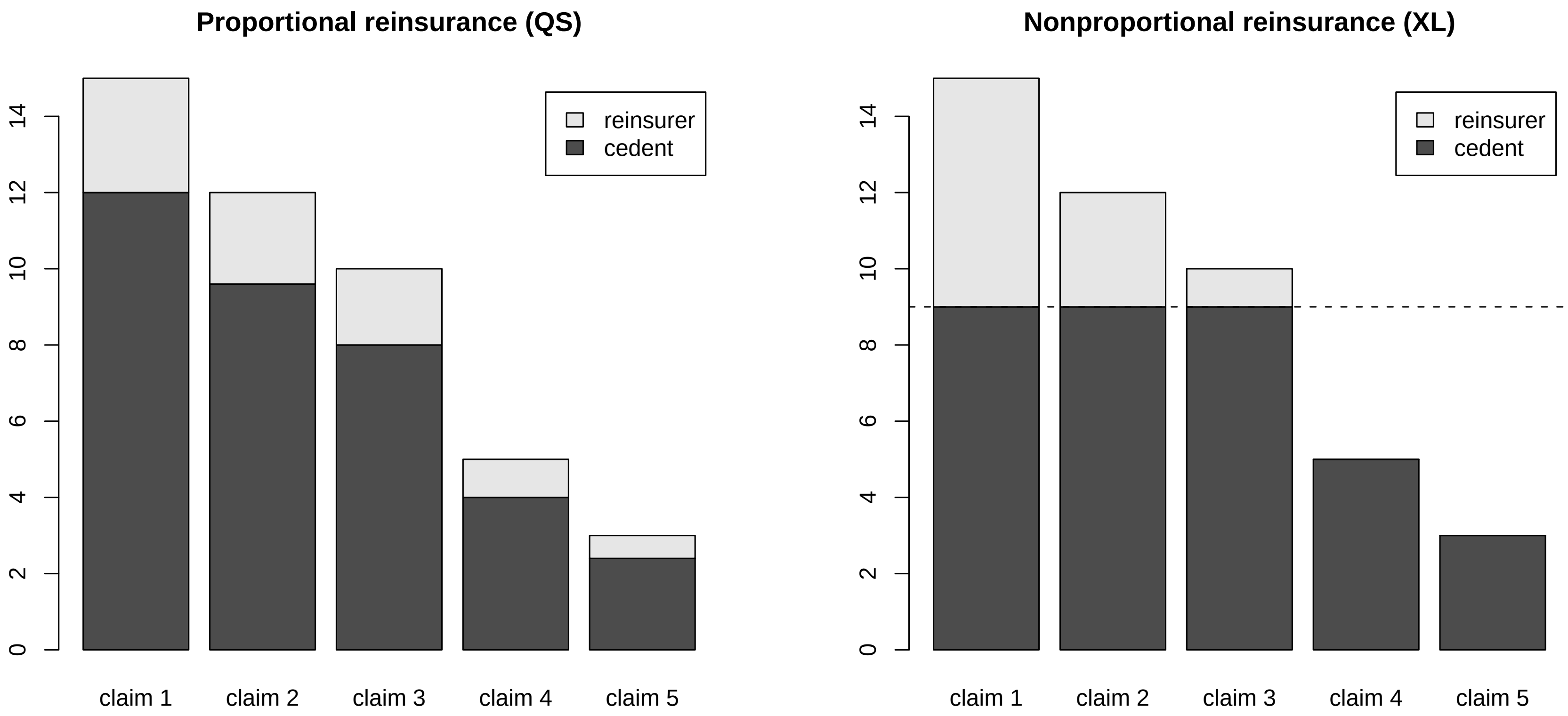

Fig. 1 – Reinsurance mechanism for claims indemnity, proportional versus nonproportional treaties.

Mathematical framework

Classical Cramér-Lundberg framework :

- claims arrival is driven by an homogeneous Poisson process, $N_t \sim \mathcal{P}(\lambda t)$, durations between consecutive arrivals $T_{i+1} - T_i$ are independent $\mathcal{E}(\lambda)$,
- claims size $X_1, \cdots, X_n, \cdots$ are i.i.d. non-negative random variables, independent of claims arrival.

Let $Y_t = \sum_{i=1}^{N_t} X_i$ denote the aggregate amount of claims during period $[0, t]$.

Premium

The pure premium required over period $[0, t]$ is

$$\pi_t = \mathbb{E}(Y_t) = \mathbb{E}(N_t)\mathbb{E}(X) = \underbrace{\lambda\mathbb{E}(X)}_{\pi} t.$$

Note that more general premiums can be considered, e.g.

- safety loading proportional to the pure premium, $\pi_t = [1 + \lambda] \cdot \mathbb{E}(Y_t)$,
- safety loading proportional to the variance, $\pi_t = \mathbb{E}(Y_t) + \lambda \cdot Var(Y_t)$,
- safety loading proportional to the standard deviation, $\pi_t = \mathbb{E}(Y_t) + \lambda \cdot \sqrt{Var(Y_t)}$,
- entropic premium (exponential expected utility) $\pi_t = \dfrac{1}{\alpha} \log\left(\mathbb{E}(e^{\alpha Y_t})\right)$,
- Esscher premium $\pi_t = \dfrac{\mathbb{E}(X \cdot e^{\alpha Y_t})}{\mathbb{E}(e^{\alpha Y_t})}$,
- Wang distorted premium $\pi_t = \displaystyle\int_0^\infty \Phi\left(\Phi^{-1}\left(\mathbb{P}(Y_t > x)\right) + \lambda\right) dx$,

A classical solvency problem

Given a ruin probability target, e.g. 0.1%, on a give, time horizon T, find capital u such that,

$$\begin{aligned}\psi(T,u) &= 1-\mathbb{P}(u+\pi t \geq Y_t, \forall t \in [0,T]) \\ &= 1-\mathbb{P}(S_t \geq 0 \forall t \in [0,T]) \\ &= \mathbb{P}(\inf\{S_t\} < 0) = 0.1\%,\end{aligned}$$

where $S_t = u + \pi t - Y_t$ denotes the insurance company surplus.

S&P Rating	Moody's Equivalent	Default Probability (Subsequent year)	Coverage Level
AAA	Aaa	0.01%	99.99%
AA	Aa3/A1	0.03%	99.97%
A	A2/A3	0.11%	99.89%
BBB	Baa2	0.30%	99.70%
BB	Ba1/Ba2	0.81%	99.19%
B	Ba3/B1	2.21%	97.79%
CCC	B2/B3	6.00%	94.00%
CC	B3/Caa	11.68%	88.32%
C	Caa/Ca	16.29%	83.71%

Source: Bank of America

A classical solvency problem

After reinsurance, the net surplus is then

$$S_t^{(\theta)} = u + \pi^{(\theta)} t - \sum_{i=1}^{N_t} X_i^{(\theta)},$$

where $\pi^{(\theta)} = \mathbb{E}\left(\sum_{i=1}^{N_1} X_i^{(\theta)}\right)$ and

$$\begin{cases} X_i^{(\theta)} = \theta X_i, & \theta \in [0,1], \text{ for quota share treaties,} \\ X_i^{(\theta)} = \min\{\theta, X_i\}, & \theta > 0, \text{ for excess-of-loss treaties.} \end{cases}$$

Classical answers : using upper bounds

Instead of targeting a ruin probability level, Centeno (1986) and Chapter 9 in Dickson (2005) target an upper bound of the ruin probability.

In the case of light tailed claims, let γ denote the “adjustment coefficient”, defined as the unique positive root of

$$\lambda + \pi\gamma = \lambda M_X(\gamma), \text{ where } M_X(t) = \mathbb{E}(\exp(tX)).$$

The Lundberg inequality states that

$$0 \leq \psi(T, u) \leq \psi(\infty, u) \leq \exp[-\gamma u] = \psi_{CL}(u).$$

Gerber (1976) proposed an improvement in the case of finite horizon ($T < \infty$).

Classical answers : using approximations $u \to \infty$

de Vylder (1996) proposed the following approximation, assuming that $\mathbb{E}(|X|^3) < \infty$,

$$\psi_{dV}(u) \sim \frac{1}{1+\gamma'} \exp\left(-\frac{\beta'\gamma'\mu}{1+\gamma'}\right) \text{ quand } u \to \infty$$

where

$$\gamma' = \frac{2\mu m_3}{3m_2^2}\gamma \text{ et } \beta' = \frac{3m_2}{m_3}.$$

Beekman (1969) considered

$$\psi_B(u) \frac{1}{1+\gamma}\left[1-\Gamma(u)\right] \text{ quand } u \to \infty$$

where Γ is the c.d.f. of the $\Gamma(\alpha, \beta)$ distribution

$$\alpha = \frac{1}{1+\gamma}\left(1+\left(\frac{4\mu m_3}{3m_2^2}-1\right)\gamma\right) \text{ et } \beta = 2\mu\gamma\left(m_2+\left(\frac{4\mu m_3}{3m_2^2}-m_2\right)\gamma\right)^{-1}$$

Classical answers : using approximations $u \to \infty$

Rényi - see Grandell (2000) - proposed an exponential approximation of the convoluted distribution function

$$\psi_R(u) \sim \frac{1}{1+\gamma} \exp\left(-\frac{2\mu\gamma u}{m_2(1+\gamma)}\right) \text{ quand } u \to \infty$$

In the case of subexponential claims

$$\psi_{SE}(u) \sim \frac{1}{\gamma\mu}\left(\mu - \int_0^u \overline{F}(x)\,dx\right)$$

Classical answers : using approximations $u \to \infty$

	CL	dV	B	R	SE
Exponential	yes	yes	yes	yes	no
Gamma	yes	yes	yes	yes	no
Weibull	no	yes	yes	yes	$\beta \in]0,1[$
Lognormal	no	yes	yes	yes	yes
Pareto	no	$\alpha > 3$	$\alpha > 3$	$\alpha > 2$	yes
Burr	no	$\alpha\gamma > 3$	$\alpha\gamma > 3$	$\alpha\gamma > 2$	yes

Proportional reinsurance (QS)

With proportional reinsurance, if $1-\alpha$ is the ceding ratio,

$$S_t^{(\alpha)} = u + \alpha\pi t - \sum_{i=1}^{N_t} \alpha X_i = (1-\alpha)u + \alpha S_t$$

Reinsurance can always decrease ruin probability.

Assuming that there was ruin (without reinsurance) before time T, if the insurance had ceded a proportion $1-\alpha^*$ of its business, where

$$\alpha^* = \frac{u}{u - \inf\{S_t, t \in [0,T]\}},$$

there would have been no ruin (at least on the period $[0,T]$).

$$\alpha^* = \frac{u}{u - \min\{S_t, t \in [0,T]\}} \mathbf{1}(\min\{S_t, t \in [0,T]\} < 0) + \mathbf{1}(\min\{S_t, t \in [0,T]\} \geq 0),$$

then

$$\psi(T,u,\alpha) = \psi(T,u) \cdot \mathbb{P}(\alpha^* \leq \alpha).$$

Proportional reinsurance (QS)

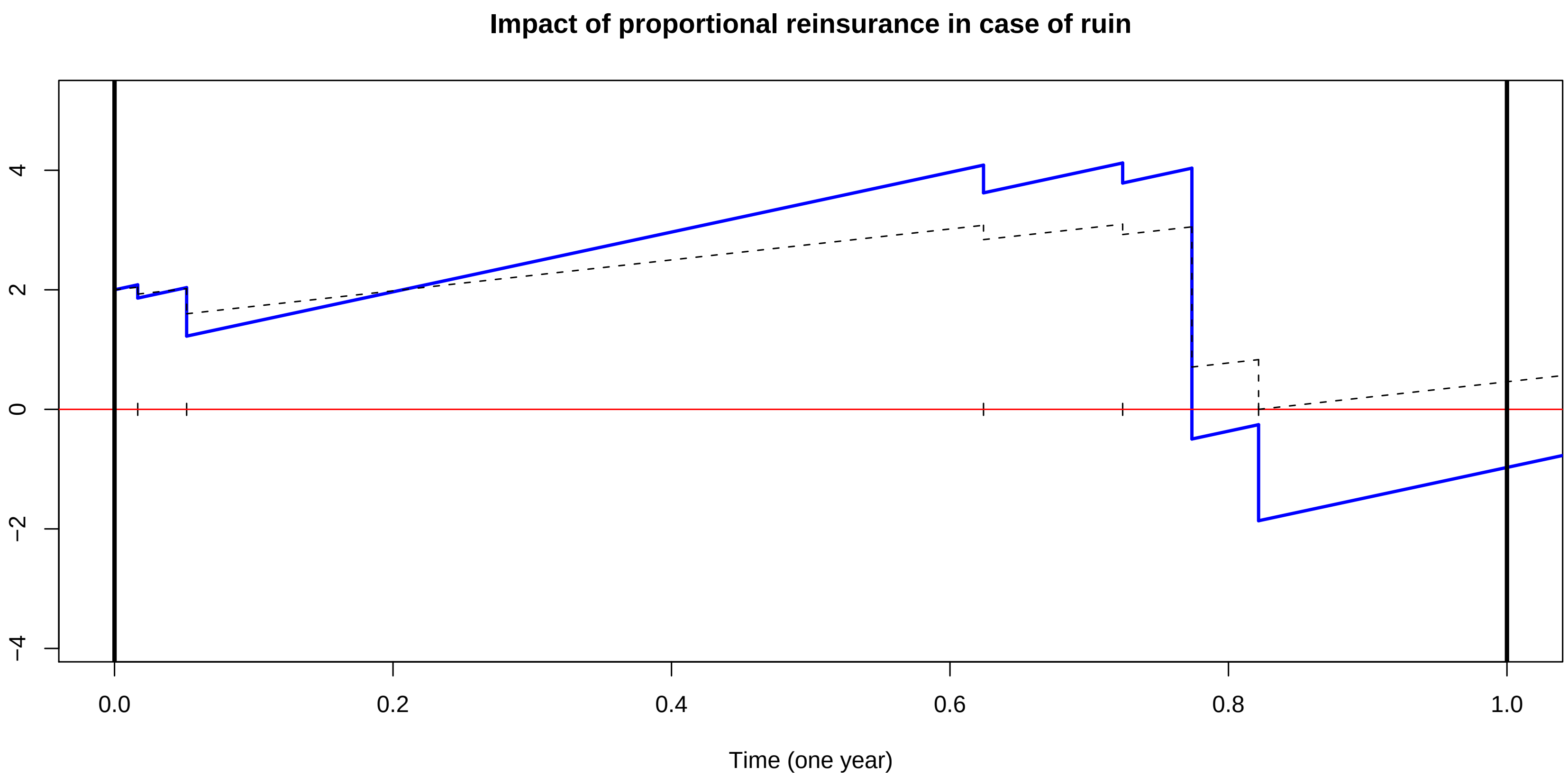

Fig. 2 – Proportional reinsurance used to decrease ruin probability, the plain line is the brut surplus, and the dotted line the cedant surplus with a reinsurance treaty.

Proportional reinsurance (QS)

In that case, the algorithm to plot the ruin probability as a function of the reinsurance share is simply the following

```
RUIN <- 0; ALPHA <- NA
for(i in 1:Nb.Simul){
  T <- rexp(N,lambda); T <- T[cumsum(T)<1]; n <- length(T)
  X <- r.claims(n); S <- u+premium*cumsum(T)-cumsum(X)
  if(min(S)<0) { RUIN <- RUIN +1
                 ALPHA <- c(ALPHA,u/(u-min(S))) }
  }
  rate <- seq(0,1,by=.01); proportion <- rep(NA,length(rate))
for(i in 1:length(rate)){
proportion[i]=sum(ALPHA<rate[i])/length(ALPHA)
}
plot(rate,proportion*RUIN/Nb.Simul)
```

Proportional reinsurance (QS)

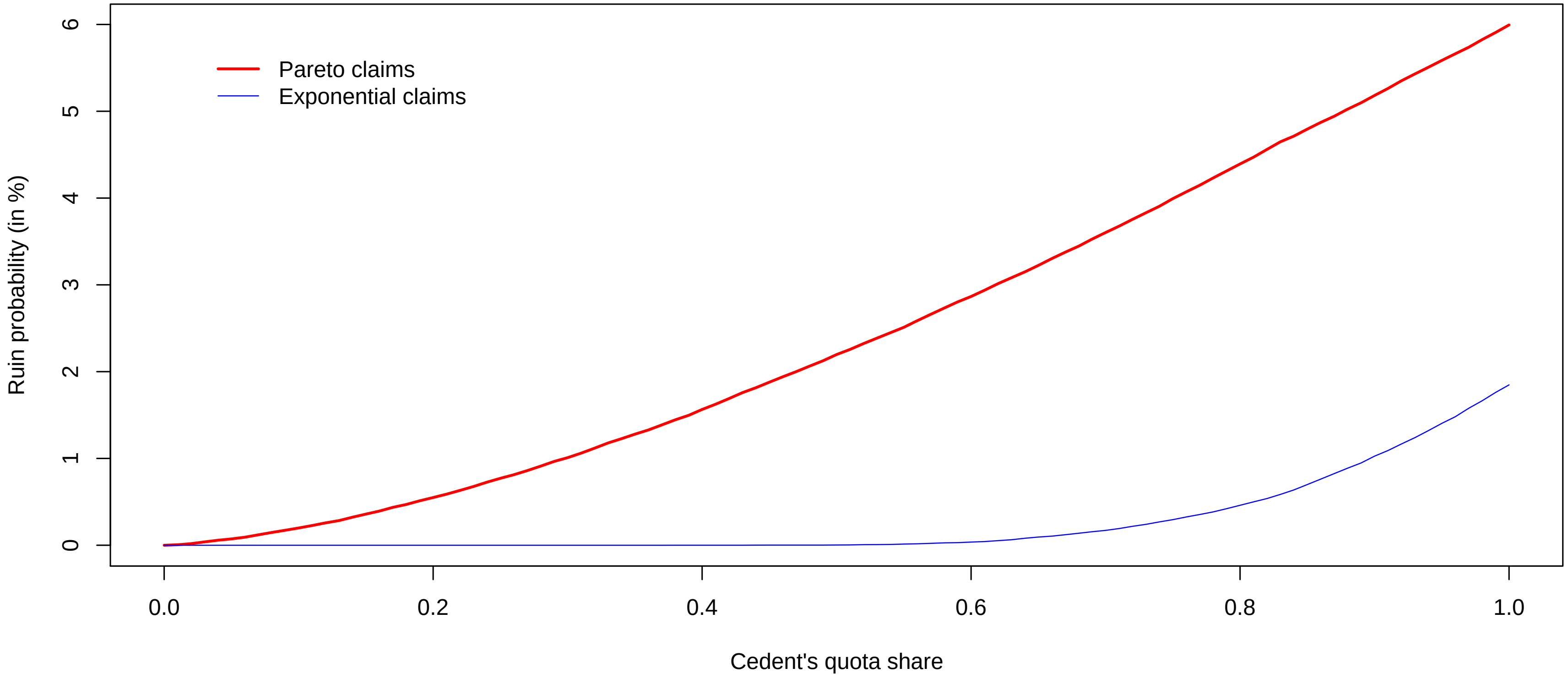

Fig. 3 – Ruin probability as a function of the cedant's share.

Proportional reinsurance (QS)

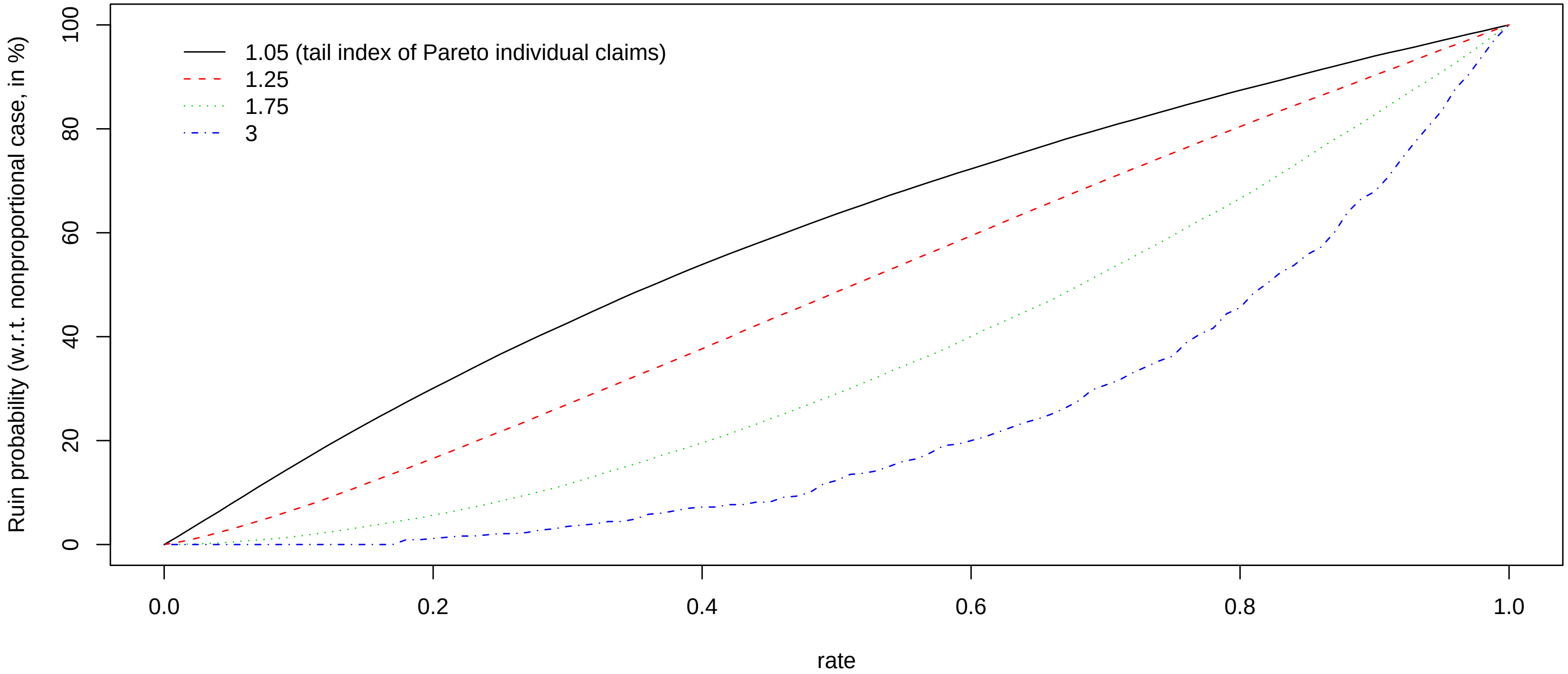

Fig. 4 – Ruin probability as a function of the cedant's share.

Nonproportional reinsurance (QS)

With nonproportional reinsurance, if $d \geq 0$ is the priority of the reinsurance contract, the surplus process for the company is

$$S_t^{(d)} = u + \pi^{(d)} t - \sum_{i=1}^{N_t} \min\{X_i, d\} \text{ where } \pi^{(d)} = \mathbb{E}(S_1^{(d)}) = \mathbb{E}(N_1) \cdot \mathbb{E}(\min\{X_i, d\}).$$

Here the problem is that it is possible to have a lot of small claims (smaller than d), and to have ruin with the reinsurance cover (since $p^{(d)} < p$ and $\min\{X_i, d\} = X_i$ for all i if claims are no very large), while there was no ruin without the reinsurance cover (see Figure 5).

Proportional reinsurance (QS)

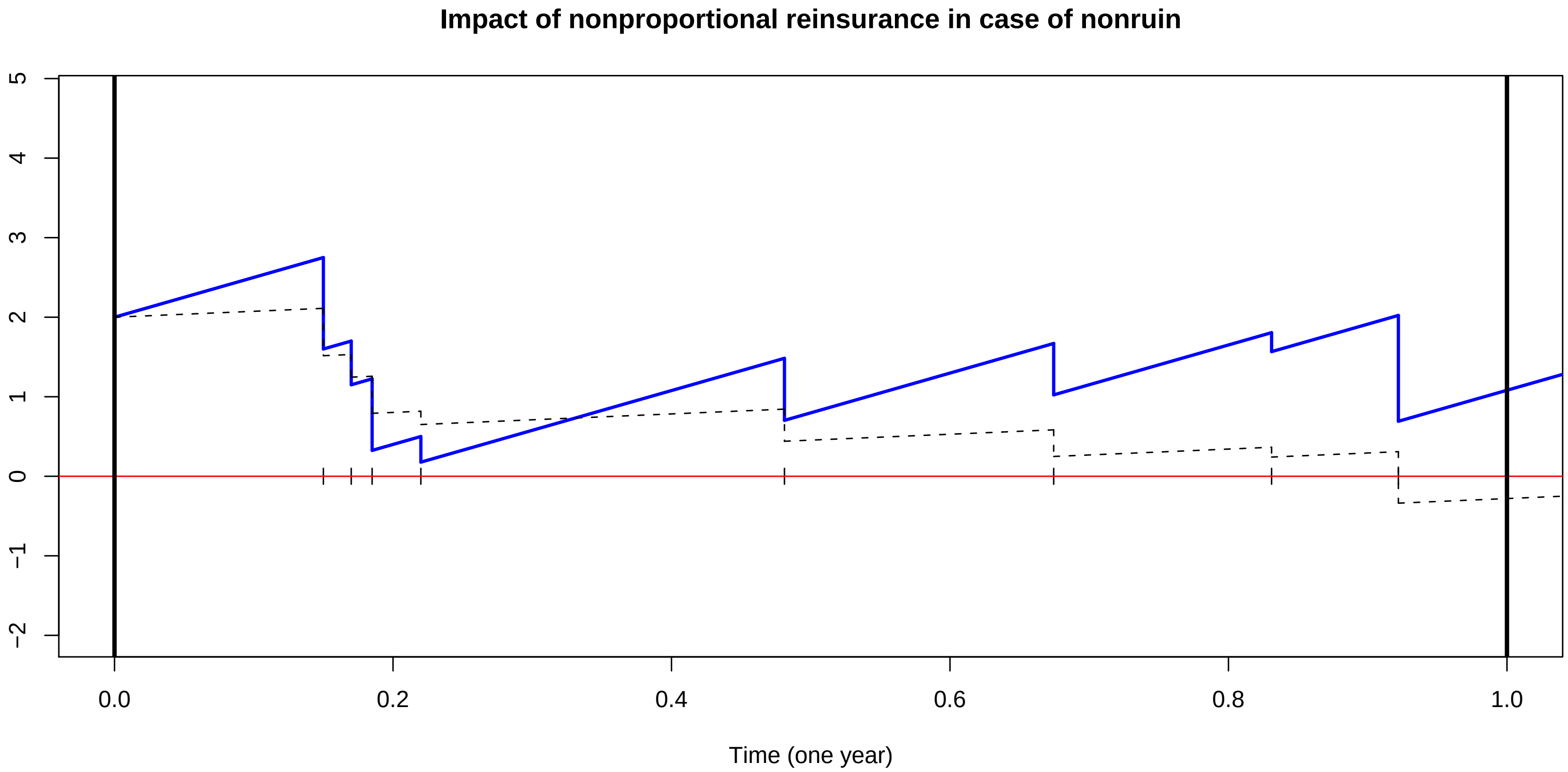

Fig. 5 – Case where nonproportional reinsurance can cause ruin, the plain line is the brut surplus, and the dotted line the cedant surplus with a reinsurance treaty.

Proportional reinsurance (QS), homogeneous Poisson

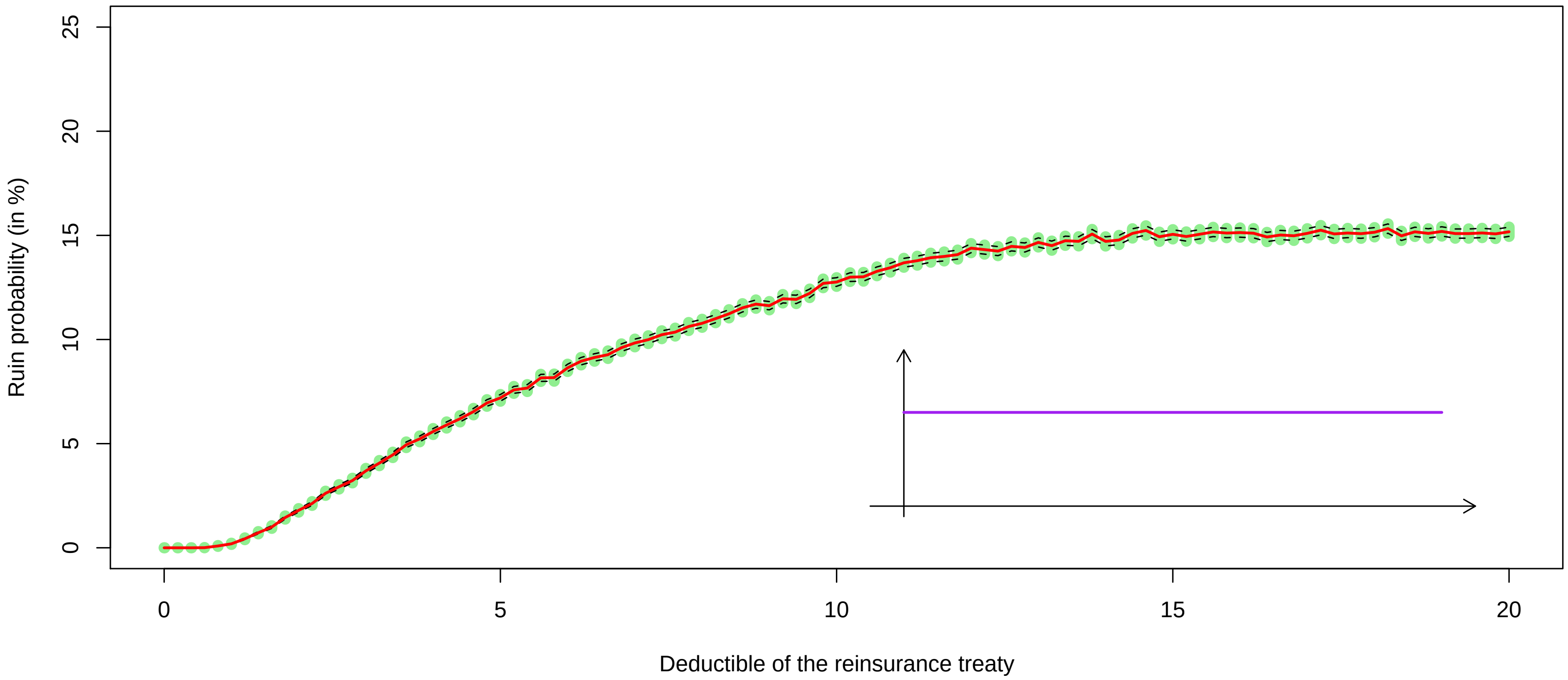

Fig. 6 – Monte Carlo computation of ruin probabilities, where $n = 100,000$ trajectories are generated for each deductible.

Proportional reinsurance (QS), nonhomogeneous Poisson

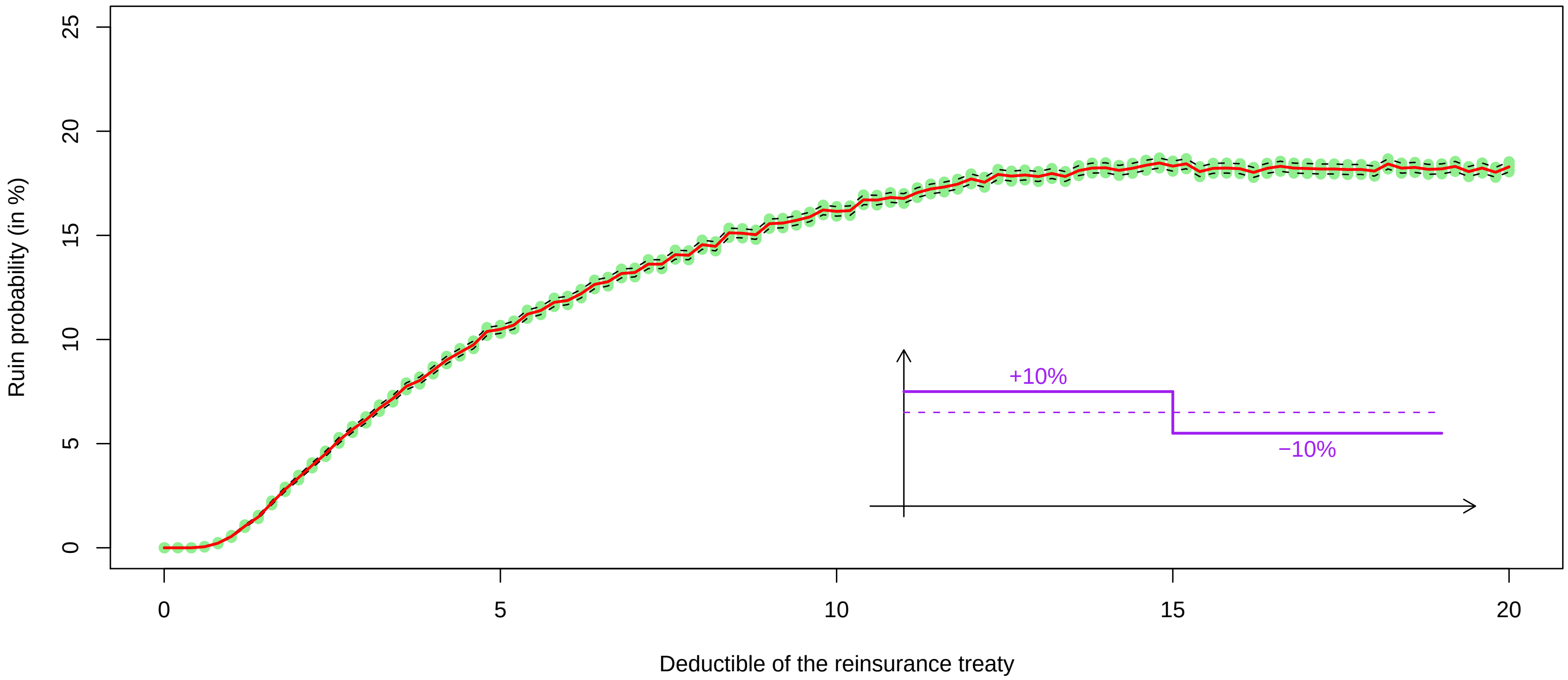

Fig. 7 – Monte Carlo computation of ruin probabilities, where $n = 100,000$ trajectories are generated for each deductible.

Proportional reinsurance (QS), nonhomogeneous Poisson

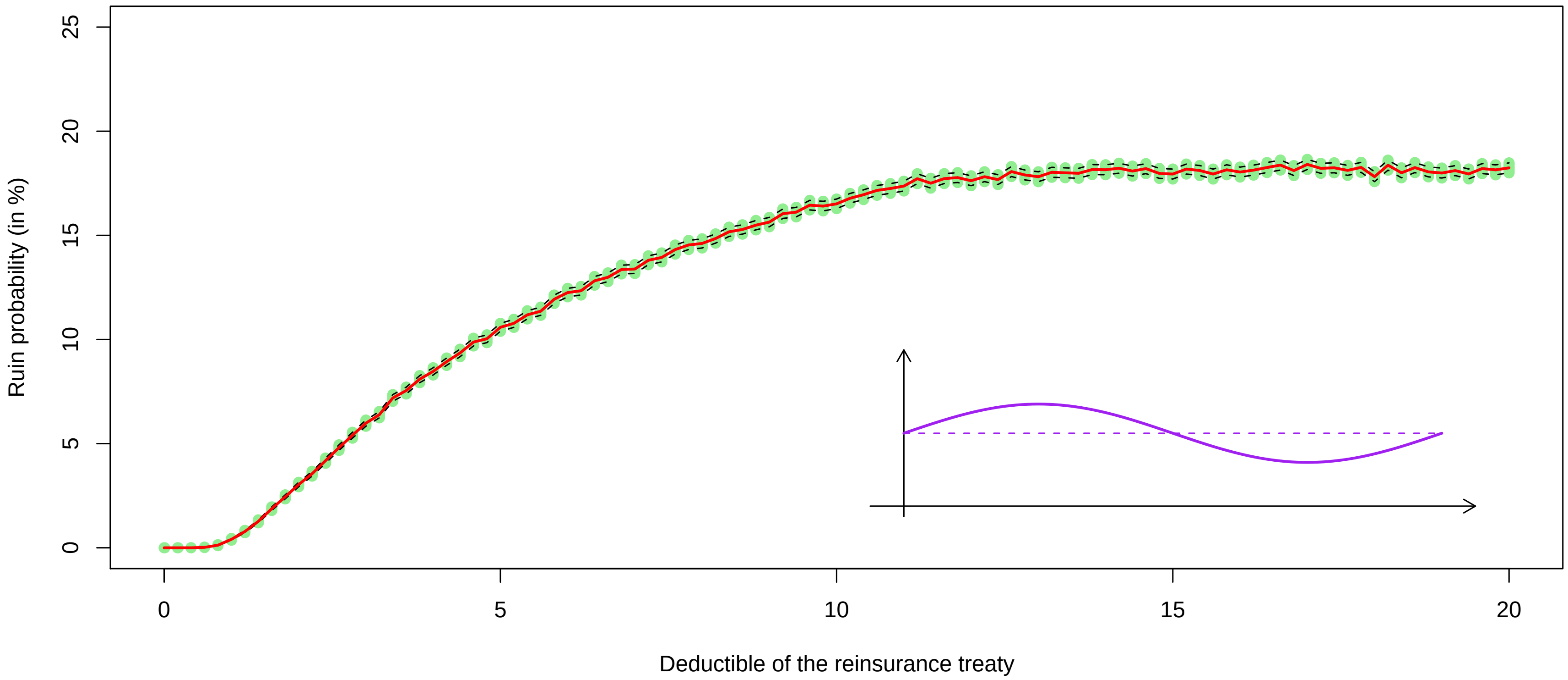

Fig. 8 – Monte Carlo computation of ruin probabilities, where $n = 100,000$ trajectories are generated for each deductible.

Proportional reinsurance (QS), nonhomogeneous Poisson

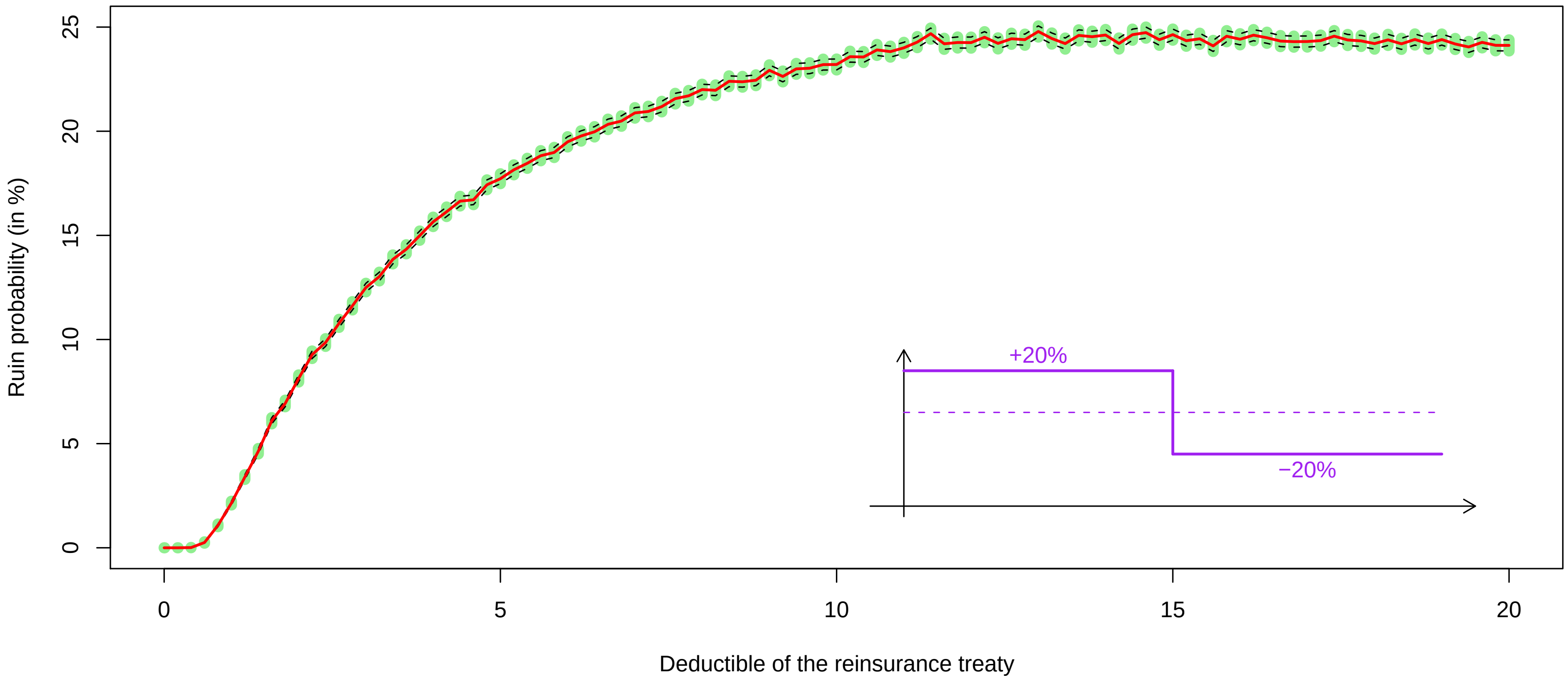

Fig. 9 – Monte Carlo computation of ruin probabilities, where $n = 100,000$ trajectories are generated for each deductible.

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